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An Extension of Schensted's Theorem

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1. INTRODUCTION

Let $\sigma = \langle a_1, a_2, \ldots, a_n \rangle$ be a sequence of integers whose terms are distinct. In 1961, C. Schensted described a method for computing the length of the longest increasing and decreasing subsequences of σ [8]. The method is based on an algorithm (discovered earlier in a somewhat different form by G. de B. Robinson [7]) which associates to σ a pair (S, T) of Young tableaux of the same shape (on n symbols) in such a way that σ can be reconstructed from S and T. If σ is a permutation of $\{1, 2, \ldots, n\}$, Schensted's algorithm gives a constructive proof of the formula

$$n! = \sum_{\lambda} (f_\lambda)^2,$$

where the sum is over all partitions λ of n and the f_λ's are the degrees of the irreducible complex representations of the symmetric group (see [6, 7]).

If σ corresponds to a tableau S whose shape is defined by a partition $\lambda = \{\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_q\}$, let $\lambda^* = \{\lambda_1^* \geqslant \lambda_2^* \geqslant \cdots \geqslant \lambda_r^*\}$ denote the partition conjugate to λ. (That is, λ_i^* is the number of parts of λ which have size at least i.) Schensted's algorithm leads to the following:

THEOREM (Schensted [8]). *The length of the longest increasing subsequence of σ is λ_1 (the number of columns of S), and the length of the longest decreasing subsequence is λ_1^* (the number of rows of S).*

The purpose of the present paper is to extend Schensted's theorem by interpreting the rest of the "shape" of λ.

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For each $k \leqslant n$, let $d_k(\sigma)$ denote the length of the longest subsequence of σ which has no increasing subsequences of length $k + 1$. It can be shown easily that any such sequence is obtained by taking the union of k decreasing subsequences. Similarly, define $a_k(\sigma)$ to be the length of the longest subsequence consisting of k ascending subsequences. Our main result is the following:

THEOREM. *For each* $k \leqslant n$,

$$a_k(\sigma) = \lambda_1 + \lambda_2 + \cdots + \lambda_k,$$

$$d_k(\sigma) = \lambda_1^* + \lambda_2^* + \cdots + \lambda_k^*.$$

While this result is naturally suggested by Schensted's theorem, it is still somewhat surprising, since we have no concrete interpretation of the individual terms λ_i and λ_i^* if $i > 1$.

The motivation for the present paper comes from a more general class of problems involving partially ordered sets. In [3], Greene and Kleitman considered subsets of a partially ordered set P which contain no chains of length $k + 1$ (or equivalently, subsets obtained by taking the union of k antichains). A subset with this property is called a *k-family* (or an *antichain* if $k = 1$). A k-family of maximum size is called a *Sperner k-family*, and its size is denoted by $d_k(P)$. If we are given a sequence $\sigma = \langle a_1, a_2, \ldots, a_n \rangle$, we can construct a partially ordered set P_σ such that $d_k(\sigma) = d_k(P_\sigma)$ for each k as follows: P_σ consists of all pairs (a_i, i), $1 \leqslant i \leqslant n$, where $(a_i, i) \leqslant (a_j, j)$ if and only if $a_i \leqslant a_j$ and $i \leqslant j$. In this ordering, chains and antichains of P_σ correspond exactly to increasing and decreasing subsequences of σ, and hence Sperner k-families correspond to maximum-sized unions of k decreasing sequences.

In [3], the structure of Sperner k-families in an arbitrary partially ordered set was studied in detail, and many of the results obtained there have useful interpretations in the present context. The main result of [3] was an extension of Dilworth's theorem for partially ordered sets [2], which shows that an exact bound for $d_k(P)$ can be obtained by looking at certain partitions of P into chains (called "k-saturated partitions"). We conclude the present paper by showing how Schensted's algorithm can be used to constructively obtain a similar result for increasing and decreasing subsequences.

2. Schensted's Algorithm

If $\lambda = \{\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_q\}$ is a partition of n, a *Young tableau* (sometimes called "standard tableau") *of shape* λ is a doubly indexed array $\{s(i,j) \mid 1 \leqslant i \leqslant q,\ 1 \leqslant j \leqslant \lambda_i\}$ such that

(i) the entries $s(i,j)$ are distinct integers, and

(ii) each row and column forms an increasing sequence.

For example, Fig. 1 shows a Young tableau of shape $\lambda = \{4, 2, 1, 1\}$ whose entries form a permutation of $\{1, 2,..., 8\}$.

$$\begin{array}{llll} 1 & 3 & 4 & 7 \\ 2 & 5 & & \\ 6 & & & \\ 8 & & & \end{array}$$

Figure 1

Let $\sigma = \langle a_1, a_2,..., a_n\rangle$ be a sequence of distinct integers. Schensted's algorithm uses σ to create an ordered pair $(S(\sigma), T(\sigma))$ of Young tableaux of the same shape, where the entries of $S(\sigma)$ are the terms of σ and the entries of $T(\sigma)$ are the integers $1, 2,..., n$. The procedure is based on an *insertion algorithm* which adds the entries $a_1,..., a_n$ successively to $S(\sigma)$. As $S(\sigma)$ is being constructed, $T(\sigma)$ is defined by setting $t(i,j) = k$ if $s(i,j)$ first becomes nonzero at the kth step. (It is easy to see that $T(\sigma)$ is a Young tableau having the same shape as $S(\sigma)$.)

Schensted's rule for "inserting" an element x is defined as follows: Suppose that $S = \{s(i,j)\}$ is a Young tableau of shape $\lambda = \{\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_q\}$ and x is an integer which does not appear in S. (1) Add x to the first row of S by inserting x in place of the least element y such that $x < y$. If no such element exists, insert x at the end of the row and stop. (2) Add y to the second row according to the same rule, and repeat the process until some element is inserted at the end of a row. Then stop.

For example, if σ is the permutation $\langle 5, 2, 3, 1, 4\rangle$, Fig. 2 illustrates the construction of $S(\sigma)$ and $T(\sigma)$ in successive steps:

$$\begin{array}{lllllllllllll} S(\sigma): & 5 & \to & 2 & \to & 2 & 3 & \to & 1 & 3 & \to & 1 & 3 \quad 4 \\ & & & 5 & & 5 & & & 2 & & & 2 & \\ & & & & & & & & 5 & & & 5 & \\ T(\sigma): & 1 & \to & 1 & \to & 1 & 3 & \to & 1 & 3 & \to & 1 & 3 \quad 5 \\ & & & 2 & & 2 & & & 2 & & & 2 & \\ & & & & & & & & 4 & & & 4 & \end{array}$$

Figure 2

Schensted's construction has many remarkable properties, and we refer the reader to [1], [4], [8], or [9] for more details and references. One of the most important properties is the following (which we will not use but include for the sake of completeness):

THEOREM 2.1 (Schensted [8], Robinson [7]). *The algorithm described above gives a one-to-one correspondence between permutations of* $\{1, 2, \ldots, n\}$ *and pairs of Young tableaux of the same shape. That is,* σ *can be reconstructed from* $S(\sigma)$ *and* $T(\sigma)$, *and if* S *and* T *are tableaux of the same shape, there exists a permutation* σ *such that* $S = S(\sigma)$, $T = T(\sigma)$.

The following result is also due to Schensted [8].

THEOREM 2.2 (Schensted). *If* σ^* *is obtained from* σ *by writing the sequence* $\langle a_1, a_2, \ldots, a_n \rangle$ *in reverse order, then* $S(\sigma^*) = S(\sigma)^T$ *(the transpose of* $S(\sigma)$*). If* $S(\sigma)$ *has shape* λ, *then* $S(\sigma^*)$ *has shape* λ^* *(the conjugate of* λ*).*

The next two results, due to Knuth [5], give a complete characterization of when two sequences σ_1 and σ_2 correspond to the same S-tableau (that is, when $S(\sigma_1) = S(\sigma_2)$). Knuth's characterization is central to the arguments which appear in Sections 3 and 4 of this paper.

THEOREM 2.3 (Knuth). *Suppose that* $x < y < z$. *Let* σ_1 *be a sequence which contains three adjacent terms of one of the following four types:* $\langle y, x, z \rangle$, $\langle y, z, x \rangle$, $\langle x, z, y \rangle$, *or* $\langle z, x, y \rangle$. *If* σ_2 *is obtained from* σ_1 *by interchanging* x *and* z, *then* $S(\sigma_1) = S(\sigma_2)$.

THEOREM 2.4 (Knuth). *Suppose* σ_1 *and* σ_2 *satisfy* $S(\sigma_1) = S(\sigma_2)$. *Then* σ_2 *can be transformed into* σ_1 *by a sequence of interchanges of the type described in Theorem* 2.3.

Theorem 2.4 is proved by showing that every sequence can be transformed (by means of the interchanges described in Theorem 2.3) into a sequence $\tilde{\sigma}$ obtained by listing the rows of $S(\sigma)$ in order starting from the bottom. It is not hard to see that sequences obtained in this way from a tableau always give the tableau back again when Schensted's algorithm is applied. Thus $\tilde{\sigma}$ can be thought of as a *canonical form* for sequences σ satisfying $S(\sigma) = S(\tilde{\sigma})$. For example, if

$$\sigma = \langle 8, 2, 6, 3, 1, 5, 4, 7 \rangle,$$

then

$$\begin{array}{llll} S(\sigma) = 1 & 3 & 4 & 7 \\ 2 & 5 & & \\ 6 & & & \\ 8 & & & \end{array}$$

FIGURE 3

Writing the rows of $S(\sigma)$ from bottom to top, we obtain

$$\tilde{\sigma} = \langle 8, 6, 2, 5, 1, 3, 4, 7 \rangle.$$

The reader can easily check that σ can be obtained from $\tilde{\sigma}$ by a valid sequence of interchanges and that $S(\sigma) = S(\tilde{\sigma})$.

To put a sequence in canonical form, one uses interchanges to duplicate the effect of the insertion algorithm. Thus transforming σ into $\tilde{\sigma}$ is no more difficult than computing $S(\sigma)$. An example of such a transformation appears in Section 4.

3. EXTENSIONS OF SCHENSTED'S THEOREM

Let $\sigma = \langle a_1, a_2, ..., a_n \rangle$ be a sequence of distinct integers, and let $\gamma = \langle a_{i_1}, a_{i_2}, ..., a_{i_q} \rangle$ be a subsequence of σ. If γ contains no increasing subsequences of length $k + 1$, we call γ a *k-decreasing subsequence of σ*. Thus "1-decreasing" is equivalent to "decreasing."

It is not hard to see that every k-decreasing subsequence is the union of k decreasing subsequences, conversely. We define a *k-increasing subsequence of σ* similarly.

Let $d_k(\sigma)$ and $a_k(\sigma)$ denote the length of the largest k-decreasing and k-increasing subsequences of σ, respectively.

THEOREM 3.1. *For each $k \leqslant n$,*

$$d_k(\sigma) = \lambda_1^* + \lambda_2^* + \cdots + \lambda_k^*,$$

$$a_k(\sigma) = \lambda_1 + \lambda_2 + \cdots + \lambda_k,$$

where $\lambda = \{\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_q\}$ is the shape of $S(\sigma)$ and $\lambda^* = \{\lambda_1^* \geqslant \lambda_2^* \geqslant \cdots \geqslant \lambda_r^*\}$ is the conjugate of λ.

Observe that Theorem 3.1 contains no interpretation of the individual terms λ_i and λ_i^*. It is *not* true that a k-decreasing sequence of size

$d_k(\sigma)$ can always be obtained by adding λ_k^* elements to a $(k-1)$-decreasing sequence of size $d_{k-1}(\sigma)$, as the following example shows:

EXAMPLE. Let $\sigma = \langle 8, 6, 3, 1, 5, 9, 7, 4, 2\rangle$. Then $d_1(\sigma) = 5$ and $d_2(\sigma) = 8$. Schensted's algorithm associates σ with the tableau

$$\begin{array}{lll} 1 & 2 & 7 \\ 3 & 4 & \\ 5 & 9 & \\ 6 & & \\ 8 & & \end{array}$$

so that $\lambda_1^* = 5$ and $\lambda_2^* = 3$, but there is only one 2-decreasing subsequence of length 8 ($\langle 8, 6, 3, 1, 9, 7, 4, 2\rangle$) and it can only be partitioned into two blocks of size four.

The proof of Theorem 3.1 consists of two observations:

(1) *The theorem holds if σ is in "canonical form." (That is, if σ is obtained by listing the rows of $S(\sigma)$ beginning with the last.)*

(2) *The values of $a_k(\sigma)$ and $d_k(\sigma)$ are unaffected by transformations of the type described in Theorem* 2.3.

By Knuth's results (Theorems 2.3 and 2.4), every sequence can be put into canonical form by means of transformations which preserve the original tableau $S(\sigma)$, and Theorem 3.1 follows immediately from these two remarks.

To prove the first observation, suppose that

$$\sigma = \langle s(q, 1), s(q, 2), \ldots, s(q, \lambda_q), \ldots, s(1, 1), \ldots, s(1, \lambda_1)\rangle,$$

where $S = (s(i, j))$ is the tableau determined by σ. We will first prove that $a_k(\sigma) = \lambda_1 + \lambda_2 + \cdots + \lambda_k$. If $k > q$, then σ itself is a k-increasing sequence, and the result is trivial. If $k \leqslant q$, then the subsequence starting with $s(k, 1)$ and containing all subsequent elements is k-increasing and has length $\lambda_1 + \lambda_2 + \cdots + \lambda_k$. To see that no larger k-increasing subsequence exists, observe that σ can be partitioned into r *decreasing* subsequences:

$$\langle s(\lambda_1^*, 1), s(\lambda_1^* - 1, 1), \ldots, s(1, 1)\rangle$$

$$\langle s(\lambda_2^*, 2), s(\lambda_2^* - 1, 2), \ldots, s(1, 2)\rangle$$

$$\langle s(\lambda_r^*, r), s(\lambda_r^* - 1, r), \ldots, s(1, r)\rangle.$$

Here, $r = \lambda_1$ is the number of columns of $S(\sigma)$ and the decreasing subsequences are obtained by reading each column from bottom to top. Since a k-increasing sequence can intersect a decreasing sequence at most k times, it follows that $a_k(\sigma)$ is at most

$$\sum_{i=1}^{r} \min\{k, \lambda_i^*\}.$$

But an easy calculation (based on the definition of λ^*) shows that this is equal to $\lambda_1 + \lambda_2 + \cdots + \lambda_k$.

To prove that $d_k(\sigma) = \lambda_1^* + \cdots + \lambda_k^*$, observe that the union of the first k decreasing sequences in the above list is a subsequence whose total length is $\lambda_1^* + \cdots + \lambda_k^*$, and hence $d_k(\sigma) \geqslant \lambda_1^* + \cdots + \lambda_k^*$. To prove the opposite inequality, observe that the rows of $S(\sigma)$ partition σ into increasing subsequences, and hence $d_k(\sigma)$ can be at most

$$\sum_{i=1}^{q} \min\{k, \lambda_i\} = \lambda_1^* + \lambda_2^* + \cdots + \lambda_k^*.$$

Remark. In general, we call a partition of σ into increasing (or decreasing) subsequences which gives an exact bound on $d_k(\sigma)$ (or $a_k(\sigma)$) a *k-saturated partition of* σ. It is not obvious that k-saturated partitions always exist, but this can be proved from general results about partially ordered sets obtained by Greene and Kleitman [3]. A special property of permutations in canonical form not shared by all permutations is the existence of partitions which are simultaneously k-saturated for all k. We will return to this question in Section 4.

To verify the second observation, it suffices to prove the following: if $x < y < z$ and either $\langle y, z, x\rangle$ or $\langle z, x, y\rangle$ appears as a sequence of three adjacent terms in σ, then switching x and z does not increase $a_k(\sigma)$. (Clearly, it cannot decrease $a_k(\sigma)$). The other two cases occurring in Theorem 2.3 follow by symmetry. Clearly, once we have proved that $a_k(\sigma)$ is invariant under all interchanges, it follows that $d_k(\sigma)$ is also invariant.

Suppose first that $\langle y, z, x\rangle$ appears in σ, and let σ' be the result of switching x and z. If $a_k(\sigma') > a_k(\sigma)$, then σ' contains a k-increasing subsequence (say γ) of length at least $a_k(\sigma) + 1$. Clearly, γ must contain both x and z since otherwise γ would be a k-increasing subsequence of σ. Suppose first that γ does not contain y. We can express γ as the union of k increasing subsequences of $\sigma'(\gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k)$, and it must be true that one of them (say γ_i) contains both x and z. (Otherwise switching x and z back again would yield a k-increasing subsequence of σ.) But then γ_i remains an increasing subsequence of σ' if x is removed and replaced by y, since $x < y < z$. Moreover, this operation transforms γ into a k-increasing subsequence of σ. By assumption, y was not originally a member of γ, and hence this new subsequence again has length $a_k(\sigma) + 1$, which is a contradiction. Next suppose that γ contains y. If we switch x and z back again, the result must contain a decreasing subsequence (call it δ) of length $k + 1$, since it would otherwise be a k-increasing subsequence of σ. In δ, both z and x must appear (with z followed by x). But then we can replace z by y in δ and the result is a decreasing subsequence of γ of length $k + 1$, contrary to the assumption that γ was a k-increasing sequence. This completes the proof of the first case. The proof of the second case (if $\langle z, x, y\rangle$ appears in σ) is similar.

4. Construction of Subsequences and Partitions

This section concerns the problem of finding k-increasing and k-decreasing subsequences of maximum size, and constructing k-saturated partitions. Recall from Section 3 that a partition of σ into decreasing subsequences of lengths $l_1, l_2, \ldots, l_m$ is *k-saturated* (with respect to $a_k(\sigma)$) if

$$a_k(\sigma) = \sum_{i=1}^{m} \min\{k, l_i\}.$$

If $k = 1$, Schensted's algorithm gives a simple method for constructing longest increasing subsequences and 1-saturated partitions: define γ_i $(1 \leqslant i \leqslant \lambda_1)$ to be the subsequence of σ consisting of all entries which "pass through" position $s(1, i)$ of the tableau. Clearly, each subsequence γ_i is decreasing. Since λ_1 is the length of the longest increasing subsequence, the collection $\gamma_1, \gamma_2, \ldots$ forms a 1-saturated partition of σ. To find an increasing subsequence of length λ_1, assign a label to each entry x of σ as follows: if x is inserted into position $s(1, i)$ when y occupies position $s(1, i - 1)$, assign x the label y. Now, if z is any element inserted in position $s(1, \lambda_1)$, an increasing subsequence of length λ_1 can be traced back from z using the labels assigned.

For $k > 1$, the situation is more complicated, and we know of no algorithm which is an elementary as Schensted's.

As we have already observed in Section 3, it is easy to solve both problems if σ is in canonical form: a maximum-sized k-increasing subsequence can be obtained by taking the largest k rows of $S(\sigma)$, while the columns of $S(\sigma)$ form a k-saturated partition (for all k). If σ is arbitrary, the method is as follows:

(1) transform σ into its canonical form, $\tilde{\sigma}$,

(2) construct a longest k-increasing subsequence of $\tilde{\sigma}$,

(3) transform $\tilde{\sigma}$ back into σ, making modifications at each step, in such a way that a longest k-increasing subsequence of σ is obtained when the transformation is complete.

In order to give the details of the proper modifications in step 3, we classify the four types of transformations occurring in Theorem 2.3 as follows:

$$\textit{Type } 1\colon \langle y, x, z\rangle \to \langle y, z, x\rangle$$

$$\textit{Type } 2\colon \langle x, z, y\rangle \to \langle z, x, y\rangle$$

$$\textit{Type } 3\colon \langle y, z, x\rangle \to \langle y, x, z\rangle$$

$$\textit{Type } 4\colon \langle z, x, y\rangle \to \langle x, z, y\rangle$$

Suppose that σ is a permutation and that σ' is obtained from σ by one of the above transformations. If γ is a k-increasing subsequence of σ of length $a_k(\sigma)$, we will describe how to modify γ in order to obtain a k-increasing subsequence γ' of σ' having the same length. We can assume that γ is partitioned into increasing subsequences $\gamma_1, \gamma_2, \ldots, \gamma_k$. It is convenient to introduce the following notation: if $w \in \gamma_i$, then $[\gamma_i, w]$ denotes the segment of γ_i which precedes (and includes) w. Similarly, $[w, \gamma_i]$ denotes the segment which follows (and includes) w.

The sequence γ' is constructed from γ according to the following rules:

(1) If σ' is obtained by a transformation of type 3 or 4 (in which a decreasing pair is switched to an increasing pair), then $\gamma_i' = \gamma_i$ for all i, and $\gamma' = \gamma$.

(2) If x and z appear in different subsequences of γ_i, then $\gamma_i' = \gamma_i$ for all i, and $\gamma' = \gamma$.

(3) If the transformation is of type 1 or 2, and $\langle x, z\rangle$ appears in some subsequence, say γ_i, then the rules depend on whether or not y occurs in γ, and are given by the following table:

	Type 1	Type 2
$y \notin \gamma$	$\gamma_p' = \gamma_p \, \forall p \neq i$ $\gamma_i' = \gamma_i - x \cup y$	$\gamma_p' = \gamma_p \, \forall p \neq i$ $\gamma_i' = \gamma_i - z \cup y$
$y \in \gamma_j$	$\gamma_p' = \gamma_p \, \forall p \neq i, j$ $\gamma_i' = \gamma_i - [\gamma_i, x] \cup [\gamma_j, y]$ $\gamma_j' = \gamma_j - [\gamma_j, y] \cup [\gamma_i, x]$	$\gamma_p' = \gamma_p \, \forall p \neq i, j$ $\gamma_i' = \gamma_i - [z, \gamma_i] \cup [y, \gamma_j]$ $\gamma_j' = \gamma_j - [y, \gamma_j] \cup [z, \gamma_i]$

For computational purposes, it is necessary to keep a record of how γ is partitioned into increasing sequences $\gamma_1, \gamma_2, \ldots, \gamma_k$. This record is updated by each step of the algorithm. We leave it for the reader to check that γ' as defined above is the union of k-increasing subsequences of σ', and has the same length as γ.

As an example, consider the permutation $\sigma = \langle 2, 4, 7, 9, 5, 1, 3, 6, 8 \rangle$, whose tableau is

$$S(\sigma) = \begin{matrix} 1 & 3 & 5 & 6 & 8 \\ 2 & 4 & 9 & & \\ 7 & & & & \end{matrix}$$

The canonical form for σ is $\tilde{\sigma} = \langle 7, 2, 4, 9, 1, 3, 5, 6, 8 \rangle$. In the following table, the left-hand column shows a sequence which transforms $\tilde{\sigma}$ back into σ. (In each case the three-term subsequence altered to obtain a given line is bracketed.) The second column shows the effect of each transformation on a maximum-sized 2-increasing subsequence.

$\tilde{\sigma} = \langle\ 7\ 2\ 4\ 9\ 1\ 3\ 5\ 6\ 8\ \rangle$	$\{2\ 4\ 9, 1\ 3\ 5\ 6\ 8\}$	
$\langle\ 7\ 2\ 4\,[1\ 9\ 3]\,5\ 6\ 8\ \rangle$	$\{2\ 4\ 9, 1\ 3\ 5\ 6\ 8\}$	(Type 4)
$\langle\ 7\ 2\ 4\ 1\,[3\ 9\ 5]\,6\ 8\ \rangle$	$\{2\ 4\ 9, 1\ 3\ 5\ 6\ 8\}$	(Type 4)
$\langle\ 7\ 2\ 4\ 1\ 3\,[5\ 9\ 6]\,8\ \rangle$	$\{2\ 4\ 9, 1\ 3\ 5\ 6\ 8\}$	(Type 4)
$\langle\ 7\ 2\,[1\ 4\ 3]\,5\ 9\ 6\ 8\ \rangle$	$\{2\ 4\ 9, 1\ 3\ 5\ 6\ 8\}$	(Type 4)
$\langle\ 7\ 2\ 1\,[4\ 5\ 3]\,9\ 6\ 8\ \rangle$	$\{2\ 4\ 5\ 6\ 8, 1\ 3\ 9\}$	(Type 1)
$\langle\ 7\ 2\ 1\ 4\,[5\ 9\ 3]\,6\ 8\ \rangle$	$\{2\ 4\ 5\ 9, 1\ 3\ 6\ 8\}$	(Type 1)
$\langle\ 7\,[2\ 4\ 1]\,5\ 9\ 3\ 6\ 8\ \rangle$	$\{2\ 4\ 5\ 9, 1\ 3\ 6\ 8\}$	(Type 1)
$\langle\ 7\ 2\,[4\ 5\ 1]\,9\ 3\ 6\ 8\ \rangle$	$\{2\ 4\ 5\ 9, 1\ 3\ 6\ 8\}$	(Type 1)
$\langle\ 7\ 2\ 4\,[5\ 9\ 1]\,3\ 6\ 8\ \rangle$	$\{2\ 4\ 5\ 9, 1\ 3\ 6\ 8\}$	(Type 1)
$\langle\,[2\ 7\ 4]\,5\ 9\ 1\ 3\ 6\ 8\ \rangle$	$\{2\ 4\ 5\ 9, 1\ 3\ 6\ 8\}$	(Type 4)
$\langle\ 2\,[4\ 7\ 5]\,9\ 1\ 3\ 6\ 8\ \rangle$	$\{2\ 4\ 5\ 9, 1\ 3\ 6\ 8\}$	(Type 4)
$\sigma = \langle\ 2\ 4\,[7\ 9\ 5]\,1\ 3\ 6\ 8\ \rangle$	$\{2\ 4\ 7\ 9, 1\ 3\ 6\ 8\}$	(Type 1)

Notice that although $S(\sigma)$ has rows of size 5 and 3, σ has no 2-increasing subsequence which can be partitioned into increasing subsequences γ_1 and γ_2 with $|\gamma_1| = 5$ and $|\gamma| = 3$. Hence the algorithm is forced to change at some point from one type of 2-family to the other.

Next we turn to the question of k-saturated partitions. Unfortunately, while a permutation in canonical form can be "completely saturated" (for all k simultaneously), this is not the case for arbitrary permutations. For example, if $\sigma = \langle 3, 6, 5, 2, 1, 4 \rangle$, then

$$S(\sigma) = \begin{matrix} 1 & 4 \\ 2 & 5 \\ 3 & \\ 6 & \end{matrix}$$

so that $a_1(\sigma) = 2$, $a_2(\sigma) = 4$, $a_3(\sigma) = 5$, and $a_4(\sigma) = 6$. A 1-saturated partition of σ must consist of two decreasing subsequences, and a moment's reflection shows that there is only one such partition: $\{\langle 3, 2, 1\rangle, \langle 6, 5, 4\rangle\}$. This partition is 1-saturated and 2-saturated but *not* 3-saturated. A partition which is 2-saturated and 3-saturated but *not* 1-saturated is $\{\langle 6, 5, 2, 1\rangle, \langle 3\rangle, \langle 4\rangle\}$.

It follows from general properties of partially ordered sets (see Greene and Kleitman [3]) that *for any k there exists a partition of σ into decreasing subsequences which is both k-saturated and $(k-1)$-saturated.*

To prove this directly, we could give an algorithm similar to the one described above for k-increasing subsequences. However, this is not necessary, as the following result shows:

THEOREM 4.1. *Let σ be a permutation of $\{1, 2,..., n\}$ whose associated Young tableau $S(\sigma)$ has shape $\lambda = \{\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_q\}$. If $k \leqslant q$, let $\gamma^* = \gamma_1^* \cup \gamma_2^* \cup \cdots \cup \gamma_l^*$ be a maximum-sized l-decreasing subsequence of σ, where $l = \lambda_k$. Define Π to be the partition of σ into decreasing subsequences obtained by taking $\gamma_1^*, \gamma_2^*,..., \gamma_l^*$ and the remaining elements as singletons. Then Π is k-saturated and $(k-1)$-saturated (with respect to $a_k(\sigma)$ and $a_{k-1}(\sigma)$).*

Proof. In the Ferrers diagram of λ, let A denote the rectangle bounded by row k and column l, and let B and C denote the remaining parts, as shown in Fig. 4:

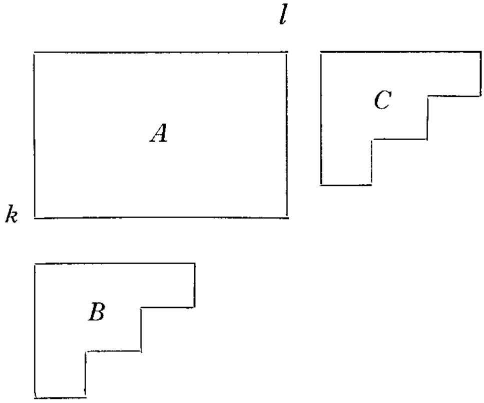

FIGURE 4

By Theorem 3.1, $a_k(\sigma) = | A | + | C |$, $a_{k-1}(\sigma) = | A | + | C | - l$, and $d_l(\sigma) = | A | + | B |$. The number of singletons in Π is exactly $| C |$, and so the bound induced on $a_k(\sigma)$ by Π is at most $kl + | C | = | A | + | C | = a_k(\sigma)$. Similarly, the bound included on $a_{k-1}(\sigma)$ is at most $(k - 1)l + | C | = | A | + | C | - \lambda_k = a_{k-1}(\sigma)$. Hence Π is both k-saturated and $(k - 1)$-saturated.

Theorem 4.1 shows that an algorithm for constructing k-saturated (and $(k - 1)$-saturated) partitions can be obtained by applying the previous algorithm to σ in reverse order.

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